Sifat-sifat Peubah Acak

Sifat-sifat Nilai Harapan

If $X = (X_1, \ldots, X_k)$ has a joint pdf $f(x_1, \ldots, x_k)$, and if $Y = u(X_1, \ldots, X_k)$ is a function of X, then $E(Y) = E_X[u(X_1, \ldots, X_k)]$, where

$$E_X[u(X_1, \ldots, X_k)] = \sum_{x_1} \cdots \sum_{x_k} u(x_1, \ldots, x_k) f(x_1, \ldots, x_k) \tag{5.2.1}$$

if X is discrete, and

$$E_X[u(X_1, \ldots, X_k)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1, \ldots, x_k) f(x_1, \ldots, x_k)\, dx_1 \ldots dx_k \tag{5.2.2}$$

if X is continuous.

If X_1 and X_2 are random variables with joint pdf $f(x_1, x_2)$, then

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

We will show this for the continuous case:

$$\begin{aligned}
E(X_1 + X_2) &= E_X(X_1 + X_2) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 + x_2) f(x_1, x_2)\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 f(x_1, x_2)\, dx_1\, dx_2 \\
&\quad + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_2 f(x_1, x_2)\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty} x_1 \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2\, dx_1 \\
&\quad + \int_{-\infty}^{\infty} x_2 \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1)\, dx_1 + \int_{-\infty}^{\infty} x_2 f_{X_2}(x_2)\, dx_2 \\
&= E_{X_1}(X_1) + E_{X_2}(X_2) \\
&= E(X_1) + E(X_2)
\end{aligned}$$

If X and Y are independent random variables and $g(x)$ and $h(y)$ are functions, then

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)] \tag{5.2.5}$$

Proof

In the continuous case,

$$\begin{aligned}
E[g(X)h(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x, y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_1(x)f_2(y)\,dx\,dy \\
&= \left[\int_{-\infty}^{\infty} g(x)f_1(x)\,dx\right]\left[\int_{-\infty}^{\infty} h(y)f_2(y)\,dy\right] \\
&= E[g(X)]E[h(Y)]
\end{aligned}$$

Covarians

The covariance of a pair of random variables X and Y is defined by

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] \tag{5.2.7}$$

Another common notation for covariance is σ_{XY}.

Teorema

If X and Y are random variables and a and b are constants, then

$$\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X, Y)$$

$$\mathrm{Cov}(X + a, Y + b) = \mathrm{Cov}(X, Y)$$

$$\mathrm{Cov}(X, aX + b) = a\,\mathrm{Var}(X)$$

Teorema

If X and Y are random variables, then

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

and $\mathrm{Cov}(X, Y) = 0$ whenever X and Y are independent.

If X_1 and X_2 are random variables with joint pdf $f(x_1, x_2)$, then

$$\operatorname{Var}(X_1 + X_2) = \operatorname{Var}(X_1) + \operatorname{Var}(X_2) + 2\operatorname{Cov}(X_1, X_2)$$

and

$$\operatorname{Var}(X_1 + X_2) = \operatorname{Var}(X_1) + \operatorname{Var}(X_2)$$

whenever X_1 and X_2 are independent.

Bukti

$$\begin{aligned}
\operatorname{Var}(X_1 + X_2) &= E[(X_1 + X_2) - (\mu_1 + \mu_2)]^2 \\
&= E[(X_1 - \mu_1) + (X_2 - \mu_2)]^2 \\
&= E[(X_1 - \mu_1)^2] + E[(X_2 - \mu_2)^2] \\
&\quad + 2E[(X_1 - \mu_1)(X_2 - \mu_2)] \\
&= \operatorname{Var}(X_1) + \operatorname{Var}(X_2) + 2\operatorname{Cov}(X_1, X_2)
\end{aligned}$$

It also can be verified that if $X_1, \ldots, X_k$ are random variables and $a_1, \ldots, a_k$ are constants, then

$$\operatorname{Var}\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i^2 \operatorname{Var}(X_i) + 2 \sum_{i<j}\sum a_i a_j \operatorname{Cov}(X_i, X_j) \tag{5.2.14}$$

and if $X_1, \ldots, X_k$ are independent, then

$$\operatorname{Var}\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i^2 \operatorname{Var}(X_i) \tag{5.2.15}$$

Korelasi

If X and Y are random variables with variances σ_X^2 and σ_Y^2 and covariance $\sigma_{XY} = \text{Cov}(X, Y)$, then the **correlation coefficient** of X and Y is

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{5.3.1}$$

Nilai Harapan Bersyarat

Definition 5.4.1

If X and Y are jointly distributed random variables, then the **conditional expectation** of Y given $X = x$ is given by

$$E(Y \mid x) = \sum_y y f(y \mid x) \quad \text{if } X \text{ and } Y \text{ are discrete} \tag{5.4.1}$$

$$E(Y \mid x) = \int_{-\infty}^{\infty} y f(y \mid x)\, dy \quad \text{if } X \text{ and } Y \text{ are continuous} \tag{5.4.2}$$

Other common notations for conditional expectation are $E_{Y|x}(Y)$ and $E(Y \mid X = x)$.

If X and Y are jointly distributed random variables, then

$$E[E(Y \mid X)] = E(Y)$$

Proof

Consider the continuous case:

$$\begin{aligned}
E[E(Y \mid X)] &= \int_{-\infty}^{\infty} E(Y \mid x) f_1(x)\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(y \mid x) f_1(x)\, dy\, dx \\
&= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x, y)\, dx\, dy \\
&= \int_{-\infty}^{\infty} y f_2(y)\, dy \\
&= E(Y)
\end{aligned}$$

If X and Y are independent random variables, then $E(Y \mid x) = E(Y)$ and $E(X \mid y) = E(X)$.

Proof

If X and Y are independent, then $f(x, y) = f_1(x) f_2(y)$, so that $f(y \mid x) = f_2(y)$ and $f(x \mid y) = f_1(x)$. In the continuous case

$$E(Y \mid x) = \int_{-\infty}^{\infty} y f(y \mid x)\, dy$$

$$= \int_{-\infty}^{\infty} y f_2(y)\, dy$$

$$= E(Y)$$

Fungsi Pembangkit Momen Bersama

Definition 5.5.1

The joint MGF of $X = (X_1, \ldots, X_k)$, if it exists, is defined to be

$$M_X(t) = E\left[\exp\left(\sum_{i=1}^{k} t_i X_i\right)\right] \tag{5.5.1}$$

where $t = (t_1, \ldots, t_k)$ and $-h < t_i < h$ for some $h > 0$.

Sifat Fungsi Pembangkit Momen

If $M_{X,Y}(t_1, t_2)$ exists, then the random variables X and Y are independent if and only if

$$M_{X,Y}(t_1, t_2) = M_X(t_1)M_Y(t_2)$$

TERIMA KASIH