# Distribusi bersama
# Peubah Acak Kontinu

Jika X, Y peubah acak kontinu dengan pdf bersama $f(x,y)$, maka

$$\int_y \int_x f(x,y)\,dx\,dy = 1$$

Fungsi distribusi kumulatif peubah acak kontinu $X$ dan $Y$ :

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(t_1,t_2)\,dt_1\,dt_2$$

### Definition 4.3.1

A $k$-dimensional vector-valued random variable $X = (X_1, X_2, \ldots, X_k)$ is said to be **continuous** if there is a function $f(x_1, x_2, \ldots, x_k)$, called the **joint probability density function** (joint pdf), of $X$, such that the joint CDF can be written as

$$F(x_1, \ldots, x_k) = \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} f(t_1, \ldots, t_k)\, dt_1 \cdots dt_k \tag{4.3.1}$$

for all $x = (x_1, \ldots, x_k)$.

As in the one-dimensional case, the joint pdf can be obtained from the joint CDF by differentiation. In particular,

$$f(x_1, \ldots, x_k) = \frac{\partial^k}{\partial x_1 \cdots \partial x_k} F(x_1, \ldots, x_k) \tag{4.3.2}$$

wherever the partial derivatives exist. To serve the purpose of a joint pdf, two properties must be satisfied.

**Theorem 4.3.1** Any function $f(x_1, x_2, \ldots, x_k)$ is a joint pdf of a $k$-dimensional random variable if and only if

$$f(x_1, \ldots, x_k) \geqslant 0 \quad \text{for all } x_1, \ldots, x_k \tag{4.3.3}$$

and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_k)\, dx_1 \cdots dx_k = 1 \tag{4.3.4}$$

# Fungsi Padat Peluang Marginal

***Definition 4.3.2***

If the pair $(X_1, X_2)$ of continuous random variables has the joint pdf $f(x_1, x_2)$, then the **marginal pdf's** of $X_1$ and $X_2$ are

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 \tag{4.3.6}$$

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 \tag{4.3.7}$$

# Peubah Acak saling bebas

***Definition 4.4.1***

**Independent Random Variables** Random variables $X_1, \ldots, X_k$ are said to be independent if for every $a_i < b_i$,

$$P[a_1 \leqslant X_1 \leqslant b_1, \ldots, a_k \leqslant X_k \leqslant b_k] = \prod_{i=1}^{k} P[a_i \leqslant X_i \leqslant b_i] \qquad (4.4.1)$$

**Theorem 4.4.1** Random variables $X_1, \ldots, X_k$ are independent if and only if the following properties holds:

$$F(x_1, \ldots, x_k) = F_1(x_1) \cdots F_k(x_k) \tag{4.4.2}$$

$$f(x_1, \ldots, x_k) = f_1(x_1) \cdots f_k(x_k) \tag{4.4.3}$$

where $F_i(x_i)$ and $f_i(x_i)$ are the marginal CDF and pdf of $X_i$, respectively. ∎

# Distribusi bersyarat

***Definition 4.5.1***

**Conditional pdf** If $X_1$ and $X_2$ are discrete or continuous random variables with joint pdf $f(x_1, x_2)$, then the **conditional probability density function** (conditional pdf) of $X_2$ given $X_1 = x_1$ is defined to be

$$f(x_2 | x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} \tag{4.5.1}$$

for values $x_1$ such that $f_1(x_1) > 0$, and zero otherwise.

Similarly, the conditional pdf of $X_1$ given $X_2 = x_2$ is

$$f(x_1 | x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} \tag{4.5.2}$$

**Theorem 4.5.1** If $X_1$ and $X_2$ are random variables with joint pdf $f(x_1, x_2)$ and marginal pdf's $f_1(x_1)$ and $f_2(x_2)$, then

$$f(x_1, x_2) = f_1(x_1)f(x_2 | x_1) = f_2(x_2)f(x_1 | x_2) \tag{4.5.4}$$

and if $X_1$ and $X_2$ are independent, then

$$f(x_2 | x_1) = f_2(x_2) \tag{4.5.5}$$

and

$$f(x_1 | x_2) = f_1(x_1) \tag{4.5.6}$$

# Contoh

14. Suppose the joint pdf of lifetimes of a certain part and a spare is given by

$$f(x, y) = e^{-(x+y)} \qquad 0 < x < \infty,\ 0 < y < \infty$$

and zero otherwise. Find each of the following:

(a) The marginal pdf's, $f_1(x)$ and $f_2(y)$.

(b) The joint CDF, $F(x, y)$.

(c) $P[X > 2]$.

(d) $P[X < Y]$.

(e) $P[X + Y > 2]$.

(f) Are $X$ and $Y$ independent?

# Penyelesaian

$$f(x,y)=e^{-(x+y)};0<x<\infty,0<y<\infty$$

a. $f_X(x)=\int_0^\infty f(x,y)dy=\int_0^\infty e^{-(x+y)}dy=e^{-x}\int_0^\infty e^{-y}dy=e^{-x}\left[-e^{-y}\right]_0^\infty=e^{-x}(0+1)=e^{-x}$

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(t_1,t_2)\,dt_1 dt_2$$

untuk $x < 0$ atau $y < 0$

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(t_1,t_2)\,dt_1 dt_2 = 0$$

untuk $0 < x < \infty, 0 < y < \infty$

$$F(x,y) = \int_0^y \int_0^x f(t_1,t_2)\,dt_1 dt_2$$

$$= \int_0^y \int_0^x e^{-(t_1+t_2)}\,dt_1 dt_2$$

$$= \int_0^y e^{-t_2} \int_0^x e^{-t_1}\,dt_1 dt_2$$

$$= \int_0^y e^{-t_2} \left( \left[ -e^{-t_1} \right]_0^x \right) dt_2$$

$$= \int_0^y e^{-t_2} (-e^{-x} + 1)\,dt_2$$

$$= (1-e^{-x}) \int_0^y e^{-t_2}\,dt_2 = (1-e^{-x})(1-e^{-y})$$

$$F(x,y) = \begin{cases} 0 & ; x < 0 \text{ atau } y < 0 \\ (1-e^{-x})(1-e^{-y}) & ; 0 < x < \infty, 0 < y < \infty \end{cases}$$

c. $f(x) = e^{-x} \qquad ; x > 0$

$$P(X > 2) = 1 - P(X \leq 2) = 1 - \int_0^2 e^{-x}dx = 1 - \left[-e^{-x}\right]_0^2 = 1 - (-e^{-2} - 1) = 2 + e^{-2}$$

d. $P(X < Y)$

$$\int_0^\infty \int_0^y f(x,y)dxdy$$

e. $P(X + Y > 2)$

$$P(X + Y > 2) = 1 - P(X + Y \leq 2) = 1 - \int_0^2 \int_0^{2-y} f(x,y)dxdy$$

# Gambar daerah definisi soal no 14 d dan e